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Second-Order Accurate No-Slip Conditions for Solving Problems of Incompressible Viscous Flows

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Boundary condition of the third kind (also known as Robbins condition) has been used to devise an extremely simple method of satisfying the no-slip boundary condition to second-order accuracy for solving problems of incompressible viscous flows using the vorticity-stream function form of the complete unsteady Navier–Stokes equations. It has been applied to the much-tested problem of flow past circular cylinders. Excellent agreement has been obtained with previous theoretical and experimental results despite the use of simple finite-difference techniques. The results obtained using the proposed approach have been compared with those obtained by straightforward use of the Dirichlet condition and it is easy to see that for unsteady problems the proposed technique is much superior to the conventional one. Finally, a study has been carried out to check the dependence of the present scheme on spatial discretization and the position of the outer boundary. It has been concluded that despite the "local" nature of the improvement, the proposed technique is encouragingly fast, stable, and accurate because it allows satisfaction of no-slip to a higher order of accuracy. © 1994 Academic Press, Inc.

INTRODUCTION

One of the more popular approaches used for analysing two-dimensional incompressible viscous flow problems is to solve the corresponding vorticity-stream function formulation of the complete Navier–Stokes (N-S) equations. Apart from its inherent simplicity, the approach is rather attractive as regards computer economy. However, despite its attractive advantages, the method has long been plagued with serious problems such as the satisfaction of proper boundary conditions. Attempts to determine realistic, accurate, and stable methods have been found to be highly frustrating. It has also found that the adequacy of any boundary condition, as determined by computational experiments, can depend upon the Reynolds number, the interior point differencing methods, the outer boundary conditions, and sometimes on initial conditions [1].

In the vorticity-stream function appraoch, two boundary conditions, namely, the stream function (ψ) and vorticity (ω) need to be specified at any boundary. Among all the boundary conditions, those at the no-slip wall are of extreme importance because it is at the no-slip wall where vorticity is produced, the subsequent convection and diffusion of which drives the flow-field. It is, however, found that whereas the stream-function field is supplied with two natural boundary conditions, none exists for the vorticity field. Generally, the value of the stream function at a no-slip boundary is taken to be zero and is used as a Dirichlet boundary condition for solving the stream-function Poisson equation. The value of the boundary vorticity is obtained by satisfying the no-slip at the body boundary to a certain degree of accuracy and by using it as a boundary condition for solution of the vorticity transport equation. Thus, the no-slip is satisfied implicitly through of the evaluation vorticity at the wall which is used for solving the vorticity transport equation.

It is, however, confusing to note that straightforward application of the above approach leads to a streamfunction field where the no-slip condition does not seem to be satisfied to an acceptable order of accuracy. For example, if second-order accurate forward-difference approximations of the no-slip condition are evaluated at the body boundary, they lead to surprisingly different values from what follows from the condition of the no-slip [1]. Although it is true that a second-order approximation to the no-slip condition should be different from the ideal no-slip if the third-order derivatives themselves are large, such a consolation does not seem to satisfy many of the investigators [2–4]. In their work, they have tried to satisfy the no-slip condition to the

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second-order accuracy as well as possible. The experience of the present authors also seem to suggest that it is indeed worthwhile to make such an attempt, especially, when unsteady solutions are important.

Most of the previous authors had adopted some kind of iterative scheme [2] in order to attain their goal. Such iterations implied a substantial increase in the computer expenses. Moreover, as has been observed in [4], these iterative schemes did not work well enough for solving unsteady problems. Thus, in the present work, a new approach has been proposed to satisfy the no-slip boundary condition to second-order accuracy. A numerical equivalent of the boundary condition of the third kind, which is also known as Robbins condition, has been derived by satisfying the no-slip condition to second-order accuracy. This condition has been used as the body boundary condition for solving the stream-function Poisson equation. In order to remain consistent, a new second-order accurate expression for evaluating the no-slip boundary vorticity has also been developed and used for solving the vorticity transport equation.

In the present work, the developed approach has been applied to solve the much-tested problem of the flow past non-rotating circular cylinders for a fairly wide range of Reynolds number ($5 < \mathrm{Re} < 10{,}000$). Besides comparing steady state values of coefficients of total drag and its components, the temporal evolution of the same and the no-slip wall vorticity has been compared with some of the available theoretical and experimental results. The agreement obtained is a fair proof of the reliability and accuracy of the simple approach developed.

The same method has been compared with the more conventional approach of applying the Dirichlet condition on the body boundary. It has been found that the present method is not only more accurate but also more stable than the conventional first-order accurate method which is clearly contrary to the popular belief that the use of a higher-order accurate expression for evaluating no-slip wall vorticity leads to numerical instabilities.

Finally, a short study has been presented to indicate the dependence of the present technique on the space discretization and the position of the outer boundary.

GOVERNING EQUATIONS

It is assumed that an infinitely long circular cylinder of radius "a," suddenly appears in an existing uniform flow of incompressible viscous fluid flowing at a speed U_∞. A coordinate system is chosen which is fixed with the circular cylinder (Fig. 1). The unsteady N-S equations in polar coordinates written in the vorticity-stream function formulation are

$$\frac{\partial\tilde{\omega}}{\partial\tilde{t}} - \frac{1}{\tilde{r}}\left(\frac{\partial}{\partial\tilde{r}}\left(\tilde{\omega}\,\frac{\partial\tilde{\psi}}{\partial\theta}\right) - \frac{\partial}{\partial\theta}\left(\tilde{\omega}\,\frac{\partial\tilde{\psi}}{\partial\tilde{r}}\right)\right) = \nu\,\nabla^2\tilde{\omega} \tag{1}$$

$$\tilde{\omega} = \nabla^2\tilde{\psi}, \tag{2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}} + \frac{1}{\tilde{r}^2}\frac{\partial^2}{\partial\theta^2},$$

where $(\tilde{r}, \theta)$ are the polar coordinates, ν is the kinematic viscosity, t is the time. The variables $\tilde{\psi}$ and $\tilde{\omega}$ are defined by

$$\tilde{u} = -\frac{1}{\tilde{r}}\frac{\partial\tilde{\psi}}{\partial\theta}, \qquad \tilde{v} = \frac{\partial\tilde{\psi}}{\partial\tilde{r}},$$

$$\tilde{\omega} = \frac{1}{\tilde{r}}\left(\frac{\partial}{\partial\tilde{r}}(\tilde{v}\tilde{r}) - \frac{\partial\tilde{u}}{\partial\theta}\right).$$

If we assume

$$r = \frac{\tilde{r}}{a} = e^{\pi\xi}, \qquad \theta = \pi\cdot\eta, \qquad t = \frac{U_\infty\cdot t}{a},$$

$$\mathrm{Re} = \frac{2\cdot U_\infty\cdot a}{\nu}, \qquad \psi = \frac{\tilde{\psi}}{U_\infty\cdot a}, \qquad \omega = \frac{\tilde{\omega}\cdot a}{U_\infty},$$

(1) and (2) can be written in the dimensionless forms,

$$\frac{\mathrm{Re}}{2}\left(g(\xi)\,\frac{\partial\omega}{\partial t} + \frac{\partial}{\partial\eta}\left(\frac{\partial\psi}{\partial\xi}\,\omega\right) - \frac{\partial}{\partial\xi}\left(\frac{\partial\psi}{\partial\eta}\,\omega\right)\right) = \nabla^2\omega \tag{3}$$

$$\nabla^2\psi = g(\xi)\cdot\omega \tag{4}$$

with $\nabla^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$, and $g(\xi) = \pi^2\cdot e^{2\pi\xi}$.

The boundary conditions for the above problems are:

(i) no-penetration and no-slip on the cylinder surface, and

(ii) zero perturbation at infinity.

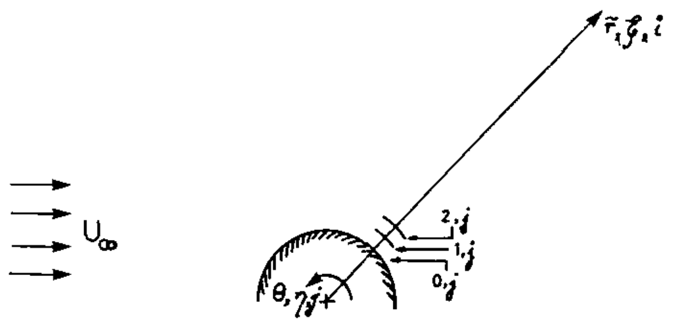

FIG. 1. The flow field axes and discretization.

BOUNDARY CONDITIONS

In order to remain contextual, we shall stick to cylindrical polar coordinates and flows past circular cylinders only. However, the derivations that follow should be extendable to any other coordinate system without much difficulty.

Let us assume that the entire flow field has been discretized in the manner shown in Fig. 1. For the present work, let us also assume that the far field boundary conditions are known to be free-stream conditions and concentrate on the no-slip boundary conditions only. It is known that these conditions are given by

$$\psi = \psi_b \quad \text{and} \quad \partial\psi/\partial\xi = \Omega_b \tag{5}$$

for $\xi = \xi_{\text{body}}$ and $t \geq 0$. The first condition in (5) implies no-penetration and the second one implies no-slip.

Stream Function Boundary Condition

As remarked before, the value of the stream function at a no-slip boundary is generally taken to be zero and is used as a Dirichlet boundary condition for solving the Poisson equation. This approach has its drawbacks, the principal one being insufficient satisfaction of no-slip. The methods that have been devised to circumvent this problem are computationally rather expensive and not reliable for unsteady calculation [1]. Thus, in the present work the main aim has been to devise a method such that (i) the second-order approximation to the no-slip remains properly satisfied throughout the calculation, (ii) is computationally inexpensive, and (iii) is reliable and accurate enough to perform unsteady calculations.

The natural candidate for satisfying condition (i) is the Neumann boundary condition which is well known for its slow rate of convergence, and thus contradicts (ii). Moreover, satisfaction of the Neumann condition guarantees a solution unique only up to an additive constant. Regarding the third condition it is not possible to say anything beforehand but an educated guess suggests that if (i) is properly satisfied then (iii) will follow automatically.

Keeping the above in mind, the boundary condition that has been proposed in this work is, in fact, a combination of both the Dirichlet and Neumann boundary conditions, i.e., a Robbins boundary condition. The condition has been derived as follows.

The second-order accurate forward-difference approximation of $(\partial\psi/\partial\xi)_{0,j}$ can be written as

$$\left(\frac{\partial\psi}{\partial\xi}\right)_{0,j} = \frac{-3\psi_{0,j} + 4\psi_{1,j} - \psi_{2,j}}{2(\Delta\xi)}.$$

Using the boundary condition given by (5), the above can be reduced to

$$\Omega_b = \frac{-3\psi_b + 4\psi_{1,j} - \psi_{2,j}}{2(\Delta\xi)}.$$

Thus, we obtain

$$\psi_b + \frac{2(\Delta\xi)}{3}\Omega_b = \frac{4}{3}\psi_{1,j} - \frac{1}{3}\psi_{2,j}. \tag{6}$$

It is easy to see that the above constitutes a boundary condition of the third kind and that it guarantees a unique solution because the coefficient of Ω_b is nonzero and greater than zero [5]. Rearranging the above we obtain

$$\psi_{1,j} = \tfrac{1}{4}[3\psi_b + \psi_{2,j} + 2(\Delta\xi)\,\Omega_b] \tag{7}$$

and this is the form in which (6) has been used in the proposed scheme for solving the Poisson equation.

One interesting point to note here is that the boundary condition (7) is also suitable for solving problems of flow past moving bodies. It has recently been applied to the interesting case of flow past rotating circular cylinders and seems to be yielding good results.

Another important point that should be noted is that the use of (7) implies that the governing equation (4) cannot be explicitly satisfied at the points adjacent to the no-slip boundary. This certainly is a drawback of the proposed scheme despite the observation by several previous investigators [6] that a local error does not necessarily deteriorate the overall result. However, it has been observed that the residues at the abovesaid points become smaller as the grid is made finer and as the calculation proceeds. Thus, as will be clear in the latter part of this work, the results obtained using the proposed scheme are more accurate and reliable than the more conventional scheme, where (4) is satisfied explicitly but no-slip is not.

Expression for Evaluating No-Slip Boundary Vorticity

As has been observed before, at the no-slip boundary, the vorticity does not have any natural boundary condition. In general, the value of vorticity at the body boundary is evaluated from (4) using the no-slip condition at the body. This seems to be a fairly logical approach because, after all, it is the no-slip at the body which gives rise to the vorticity in the flow field. Thus,

$$\omega_{0,j} = \frac{1}{g_0}\left[\left(\frac{\partial^2\psi}{\partial\eta^2}\right)_{0,j} + \left(\frac{\partial^2\psi}{\partial\xi^2}\right)_{0,j}\right]. \tag{8}$$

The first term in the bracket on the right-hand side can in most cases, be taken to be zero and we shall assume it to be zero for our case.

Using finite-difference approximations, the right-hand side of (8) can be written to various orders of accuracy. For example, if we accept an accuracy of first-order, an expression (which has been previously extensively used by other investigators for solving problems of flow past stationary walls and found to be very stable and capable of producing second-order accurate results) for evaluating vorticity at the no-slip boundary is

$$\omega_{0,j} = \frac{1}{g_0}\left[\frac{2\psi_{1,j} - 2(\Delta\xi)\,\Omega_b - 2\psi_b}{(\Delta\xi)^2}\right]. \tag{9}$$

Other expressions for evaluating the no-slip wall vorticity can be obtained using other finite-difference approximations (see [6] for a fairly comprehensive review). In the present scheme, in order to remain consistent with the second-order accurate equation (7) used for solving the Poisson equation, we have developed and used the following expression which can be obtained by using the second-order forward-difference approximation to the term $\partial^2\psi/\partial\xi^2$ in (8) directly. Thus,

$$\omega_{0,j} = \frac{1}{g_0}\,\frac{-3\Omega_b + 4(\partial\psi/\partial\xi)_{1,j} - (\partial\psi/\partial\xi)_{2,j}}{2(\Delta\xi)}.$$

The above, using second-order accurate central-difference expressions, can be written as

$$\omega_{0,j} = \frac{1}{g_0}\,\frac{-4\psi_b + \psi_{1,j} + 4\psi_{2,j} - \psi_{3,j} - 6(\Delta\xi)\,\Omega_b}{4(\Delta\xi)^2}. \tag{10}$$

NUMERICAL IMPLEMENTATION

An extremely simple framework has been used for the numerical implementation of the above. An alternate direction implicit (ADI) conservative finite-difference scheme has been used for solving the vorticity transport equation as follows:

$$\begin{aligned}
\mathrm{Re}\,\frac{g_i}{\Delta t}\,\omega_{i,j}^{n+1/2} &- \frac{\omega_{i,j+1}^{n+1/2} - 2\omega_{i,j}^{n+1/2} + \omega_{i,j-1}^{n+1/2}}{(\Delta\eta)^2} \\
&+ \frac{\mathrm{Re}}{4}\,\frac{[(\delta\psi/\delta\xi)\omega]_{i,j+1}^{n+1/2} - [(\delta\psi/\delta\xi)\omega]_{i,j-1}^{n+1/2}}{\Delta\eta} \\
= \mathrm{Re}\,\frac{g_i}{\Delta t}\,\omega_{i,j}^{n} &- \frac{\omega_{i+1,j}^{n} - 2\omega_{i,j}^{n} + \omega_{i-1,j}^{n}}{(\Delta\xi)^2} \\
&+ \frac{\mathrm{Re}}{4}\,\frac{[(\delta\psi/\delta\eta)\omega]_{i+1,j}^{n} - [(\delta\psi/\delta\eta)\omega]_{i-1,j}^{n}}{\Delta\xi};
\end{aligned}$$

$$\begin{aligned}
\mathrm{Re}\,\frac{g_i}{\Delta t}\,\omega_{i,j}^{n+1} &- \frac{\omega_{i+1,j}^{n+1} - 2\omega_{i,j}^{n+1} + \omega_{i-1,j}^{n+1}}{(\Delta\xi)^2} \\
&+ \frac{\mathrm{Re}}{4}\,\frac{[(\delta\psi/\delta\eta)\omega]_{i+1,j}^{n+1} - [(\delta\psi/\delta\eta)\omega]_{i-1,j}^{n+1}}{\Delta\xi} \\
= \mathrm{Re}\,\frac{g_i}{\Delta t}\,\omega_{i,j}^{n+1/2} &- \frac{\omega_{i,j+1}^{n+1/2} - 2\omega_{i,j}^{n+1/2} + \omega_{i,j-1}^{n+1/2}}{(\Delta\eta)^2} \\
&+ \frac{\mathrm{Re}}{4}\,\frac{[(\delta\psi/\delta\xi)\omega]_{i,j+1}^{n+1/2} - [(\delta\psi/\delta\xi)\omega]_{i,j-1}^{n+1/2}}{\Delta\eta},
\end{aligned}$$

where "Δt" is the time-increment used, subscript "n" denotes the nth time step, and "$n+1/2$" denotes the values of the intermediate time step between "n" and "$n+1$"th time steps. The space derivatives in the above two equations have been represented by second-order accurate central-difference approximations.

A simple successive overrelaxation (SOR) has been used for solving the Poisson equation. The coupling between the two equations has been kept explicit. Besides the proposed boundary conditions (Eq. (7) for the stream function and (10) for the vorticity) the results have been obtained for comparison using the conventional boundary conditions (first equation of (5) for the stream function and (9) for the vorticity).

RESULTS AND DISCUSSIONS

The results presented below have been broadly divided into three parts. In part A, the proposed scheme has been validated by comparing it with available theoretical (both analytical and numerical in nature) and experimental results. In part B, the proposed approach has been compared with the conventional one. In part C, a study has been carried out to obtain an idea of the dependence of the present scheme on spatial resolution and the position of the outer boundary. In each part, studies have been carried out for both steady and unsteady states of the flows.

A. *Validation*

The proposed approach has been validated for both steady state solutions and the extremely critical starting phase of the flow. To start with, we consider the steady state solutions.

In Table I, the steady state drag coefficients, both the total and the components, for various Reynolds numbers as obtained using the present approach, have been presented along with other results available from the current literature [9–11]. The comparison clearly indicates that, as far as steady state solutions are concerned, the proposed method produces accurate and reliable results. It should be indicated here that while using the proposed scheme it was assumed that $\Delta\xi = 0.02$, $\Delta\eta = 2/60$, and $r_\infty = 152.4060$.

TABLE I

Comparison with Results Obtained from Ref. [9–11]

Re	Ref. [9]	Ref. [10]	Ref. [11]	Present
		Pressure component of drag		
5		2.104	2.199	2.1540
20		1.201		1.2131
		Friction component of drag		
5		1.843	1.917	1.8912
20		0.794		0.7958
		Total coefficient of drag		
5		3.947	4.116	4.0452
20	2.001	1.995	2.045	2.0189

For evaluating the present scheme for the extremely critical phase of starting flow, a fairly large range of Reynolds number (500 to 10,000) have been studied. The evolution of the coefficients of drag (total, as well as components), no-slip wall vorticity with time, and obtained stream-line patterns have been compared with results obtained by the various theoretical and experimental works.

Among the theoretical works, [12, 13] are analytical in nature. They are known to be valid for small times and moderately large Reynolds numbers. The works [7, 8, 14, 15] are numerical in nature. They are expected to be valid for a longer time and in some cases results up to $t = 6.0$ have been presented.

In Figs. 2a and b, the evolution of the drag coefficients with time has been compared with the results due to [12, 14] for Reynolds numbers 1000 and 10,000. The grid (radial $*$ angular) used for $\mathrm{Re} = 1000$ is $1000 * 60$ and for $\mathrm{Re} = 10{,}000$ it is $1500 * 60$. The outer radius in both the cases is five. In both the cases the calculations were started with a small time increment of 0.0001 in order to obtain accurate results in the extremely critical phase of the starting flow. It should be noted that if the results in the starting phase were not important, we could have used much larger time increments as discussed in the following part of Results and Discussions. From the figures it can be seen easily that the present results agree reasonably well with the analytical ones. The agreement deteriorates with time as expected because the analytical results are known to become inaccurate as time increases. An encouraging evidence of the accuracy of the present approach is the fact that the calculated drag coefficient remains positive throughout and compares well with [12]. In comparison with the results represented in [14, 15], the drag coefficient goes to negative values for $\mathrm{Re} = 10{,}000$. In addition to producing negative values of drag, the results presented in [14] are found to oscillate unlike in any other work. The oscillations obtained in the results of [14] are probably due to the use of extremely large time steps.

In Fig. 3, the time evolution of the pressure component, the friction component, and the total drag have been compared with results presented in [12] for $\mathrm{Re} = 500$. The grid used is $350 * 60$ for an outer radius of 20. The time increment in this case is 0.0005. It should be noted that, in comparison with the previous cases, the grid in the present case is extremely coarse. Despite the coarse grid, it can be seen that at small times, not only the total drag, but also its components, are in fair agreement with the analytical results. This fact testifies strongly to the reliability of the approach developed. At larger times the analytical results are known to be inaccurate and the comparison becomes poor. However, it is interesting to note that only the pressure component of the drag disagrees with the analytical results, whereas the friction component is in excellent agreement throughout.

In Fig. 4, boundary vorticity results obtained at various time steps have been compared with those in [12, 15] for

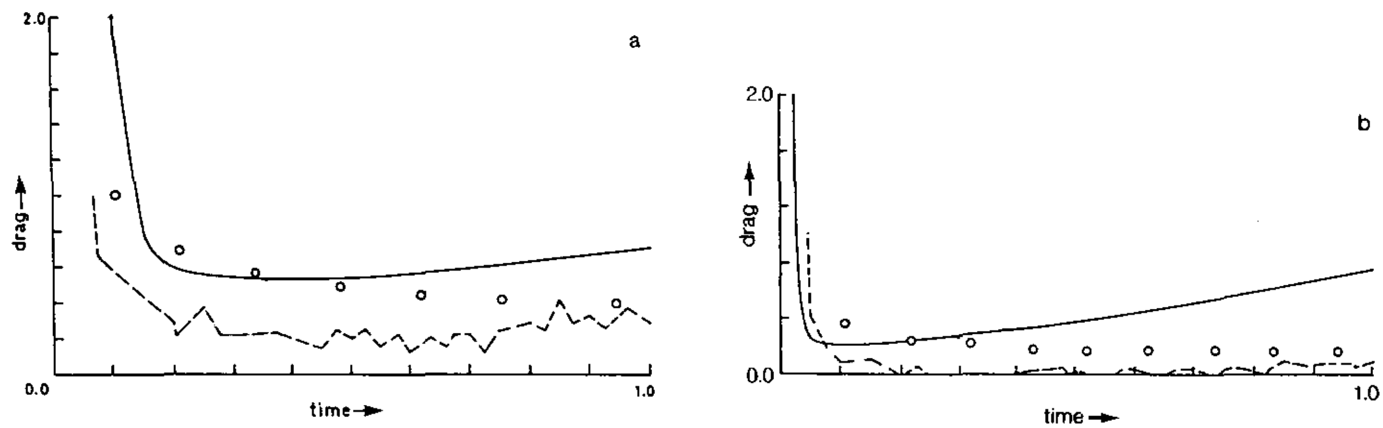

FIG. 2. Comparison of the evolution of C_d with time for various Reynolds numbers. (a) Ref. [12], Re = 1000, ○; Ref. [14], Re = 1000, ---; present results, ———; (b) Ref. [12], Re = 10,000, ○; Ref. [14], Re = 10,000, ---; present results, ———.

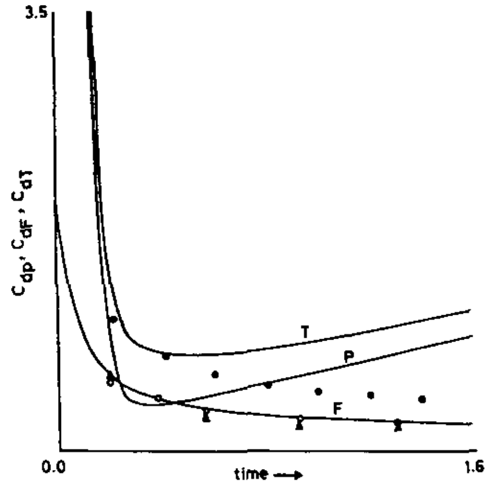

FIG. 3. Comparison of the evolution of the total drag coefficient and its components with time at Reynolds number 500. Results from Ref. [12]: ●, total drag; ○, friction component; ▲, pressure component; present results, ———.

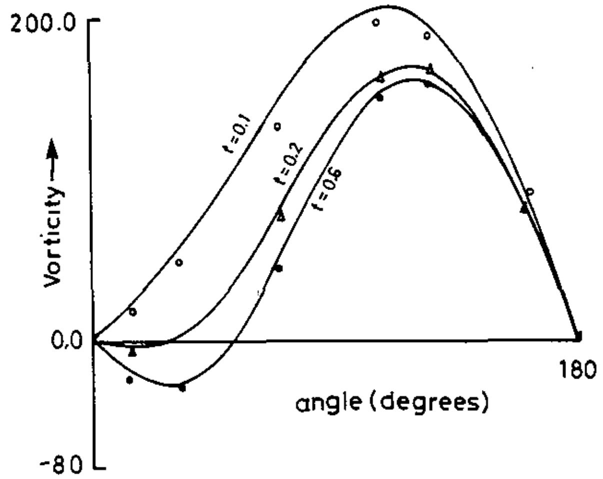

FIG. 5. Boundary vorticity distribution at Re = 10,000. Results from Ref. [12]: ○, ▲, ●; present results, ———.

Reynolds number 1000 for times 0.2 and 1.0. At the earlier time they compare fairly well with each other. However, at time $t = 1.0$, the results presented in [12] are once more found to be somewhat different, the reason being quite obvious. It is interesting to note that the numerical results remain in close agreement throughout the time span. In Fig. 5, a similar comparison has been presented for Re = 10,000 with results obtained from [12]. The agreement is found to be good up to $t = 0.6$.

In Fig. 6 and 7, examples of evolution of the streamline pattern with time have been presented. The Reynolds numbers considered are 550 and 9500. It should be noted that the results have been obtained by making the assumption of symmetry which is known to be valid shortly after the impulsive start. Qualitatively speaking, they are almost identical to the theoretical and experimental results presented in [16, 8]. It has been rather pleasing to note that even the appearance and evolution of a bulge, secondary separation, α- and β-phenomena were fairly well predicted. It was found that obtaining this qualitative similarity was much easier (especially, as regards space discretization) than to obtain the quantitative accuracy illustrated in Figs. 2 to 5.

Thus, on the basis of the above results and comparisons, it may be said that the present approach is a reasonably accurate one which produces results comparable even to

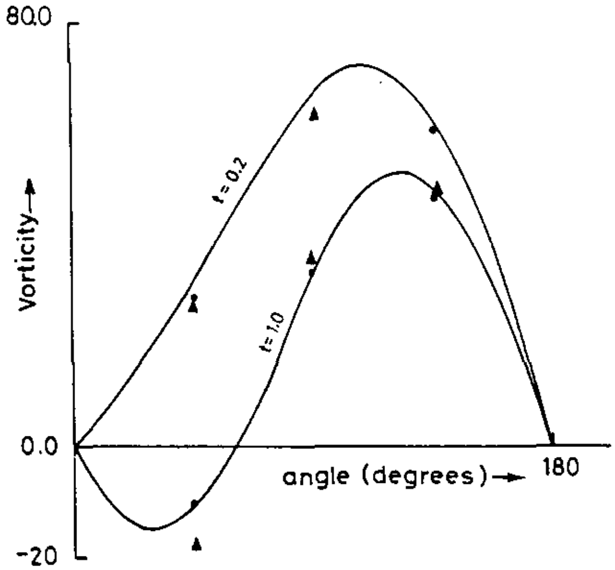

FIG. 4. Boundary vorticity distribution at Re = 1000. Results from Ref. [12], ▲; from Ref. [15], ●; present results, ———.

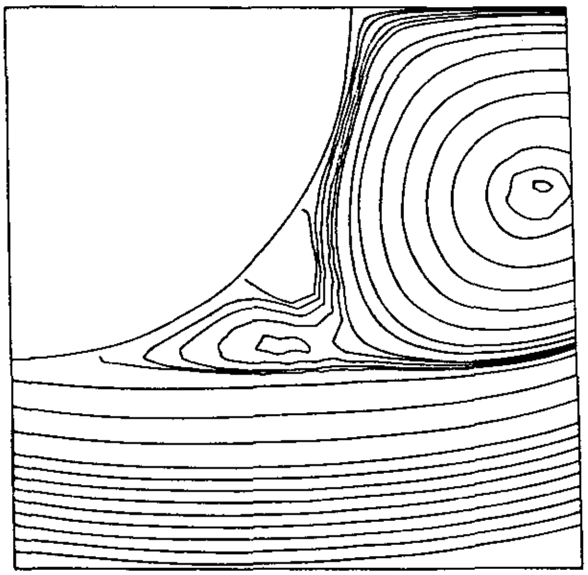

FIG. 6. A close-up view of the secondary vortex and the α-phenomenon at Re = 550, $t = 1.83$.

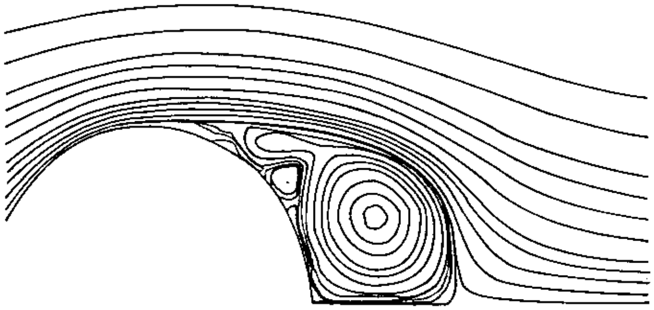

FIG. 7. Stream-function field at Re-9,500, $t = 4.01$.

TABLE II

Comparison between Schemes (C) and (P) for the Components and the Total Coefficient of Drag

	C_{dP}		C_{dF}		C_D	
Re	(C)	(P)	(C)	(P)	(C)	(P)
5	2.1518	2.1540	1.8899	1.8912	4.0417	4.0452
20	1.2078	1.2131	0.8053	0.8058	2.0131	2.0189

available analytical results for a large range of Reynolds number. It is important to note that such accuracy has been achieved without going for elaborate inner iterations as in [2] or the Hermitian technique, as in [7, 8]. Despite its utter simplicity and one drawback (as discussed earlier), the present scheme has produced results at least as good as the other theoretical methods. To the best of our knowledge, no other finite-difference scheme has been as extensively compared with other methods, analytical, numerical, and experimental for such a wide range of Reynolds number. Since only one minicomputer (of 8 MB RAM and 25 MHz speed) was available during this work, it has not been possible to extend the present method to larger Reynolds numbers. However, it seems unlikely that any problem should occur in solving similar problems as long as the flow remains laminar.

B. *Comparison*

In this part of the discussion, the conventional approach will be termed as (C) and the proposed one as (P). In Table II the steady state values of drag coefficient as obtained by the two schemes has been presented. From the obtained results it seems that the accuracy of both the schemes are comparable at steady state.

For the following study of the unsteady behaviour of the two approaches, the problem of flow past circular cylinder at Re = 100 has been considered. The grid has 175 * 120 nodes in the radial and circumferential directions, respectively, with $r_\infty = 20.658$. In Fig. 8a, the results have been obtained using $\Delta t = 0.05$. It can be seen that (P) behaves in a superior manner as regards both stability and accuracy. In Fig. 8b, $\Delta t = 0.1$, which is quite large. It can be seen that the results obtained using (C) become wildly unrealistic within a short time and do not seem to have any possibility of converging to the correct solution. However, results using (P) remain within bounds and seem to be converging to the correct solution. Thus, the proposed scheme, which uses a second-order accurate expression for evaluating the no-slip wall vorticity is found to be more accurate and dynamically far more stable than the conventional one which uses a first-order accurate expression. This is contrary to the popularly held belief [1] and confirms the superiority of the proposed scheme.

It should be pointed out here that the oscillations observed in Figs. 8a–b do not exist in reality. With a smaller time step the oscillations vanish for both the schemes. But, since the dynamic stability of the proposed scheme is superior to the conventional one, it is possible to use larger time steps for calculating an accurate unsteady evolution of the flow.

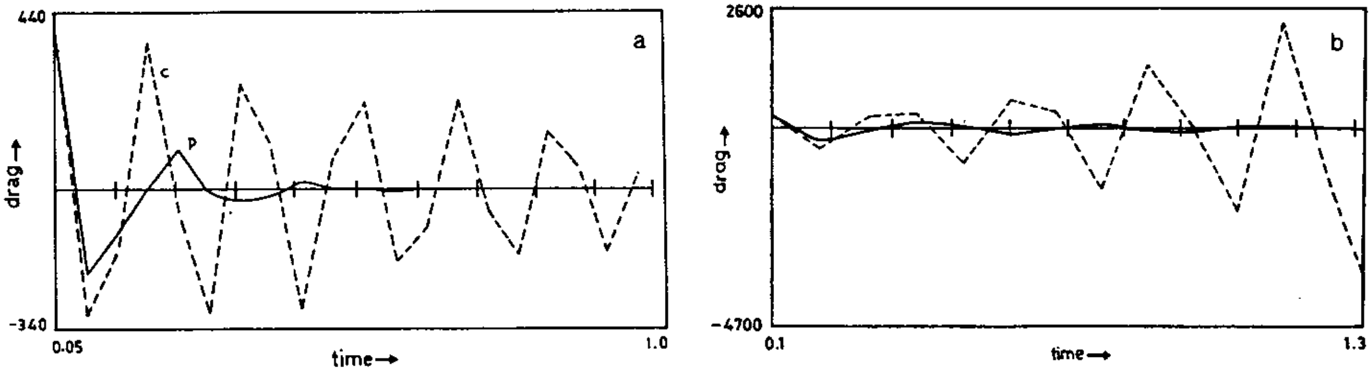

FIG. 8. Comparison of results obtained by schemes (C) and (P). The outer radius is 20.568 and the time steps are (a) 0.05; (b) 0.1; (C), ---; P, ———.

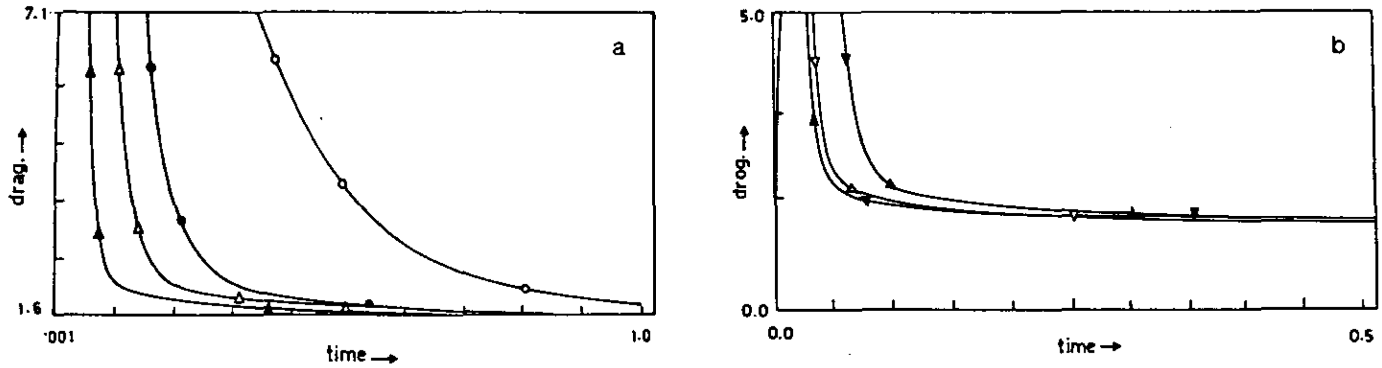

FIG. 9. Effect of spatial discretization on evolution of C_d with time. The outer radius is 6.58 and the time step is 0.05: ○, 30 nodes; ●, 60 nodes; △, 80 nodes; ▲, 120 nodes; △, 200 nodes; ▼, 300 nodes.

C. *Dependence on Spatial Discretization and Outer Boundary*

Table III presents the steady state values of drag coefficients for Reynolds number 20 as obtained by using the proposed scheme for various values of r_∞. The value of $\Delta\xi$ has been kept constant at 0.02 for all the cases. It seems clear that with the simple outer boundary conditions used, it is necessary to keep the outer boundary at a "safe" distance from the body. In this particular case, the "safe" distance seems to be around 100 radius.

For the unsteady phase, the problem is once more that of Re = 100. The outer boundary in all these cases is at $r_\infty = 6.58$. The number of nodal points in the circumferential direction is 120. The number of nodal points in the radial direction has been varied between 30 to 300. The results shown in Figs. 9a and b clearly indicate that the dependence on the spatial resolution is critical only at the early stages of the flow development. It has been observed also that the residues at the grid points adjacent to the no-slip boundary reduce as the grid is made finer. The results thus show that the discussed "drawback" of the proposed scheme does not really pose any serious threat to its reliability and accuracy.

CONCLUSIONS

Satisfaction of the no-slip to an acceptable order of accuracy has always been a cause of major concern while using the vorticity-stream function formulation for solving problems of incompressible viscous flows past stationary or moving boundaries. In the present work, a fairly simple idea has been proposed to satisfy the same to an accuracy of second-order through the use of a new set of boundary conditions while solving the Poisson and vorticity transport equations.

The proposed method has been applied to solve the problem of flow past non-rotating circular cylinders for a wide range of Reynolds number (5 < Re < 10,000). The results obtained have been compared extensively with available theoretical (both analytical and numerical) and experimental results. The comparisons seem to be extremely encouraging from both qualitative and quantitative points of view. The proposed scheme has also been found to be able to predict flow evolution of almost analytical accuracy in the extremely critical starting phase of the flow which, to say the least, is a fairly difficult task.

Comparisons with a conventional approach have made it clear that the improvements, despite being "local," lead to remarkably better results. Thus, the proposed approach has been found to be not only more accurate but also dynamically far more stable, despite the use of a higher-order accurate expression for evaluating the no-slip boundary vorticity.

A study of the dependence of the proposed method on spatial discretization has made it clear that the scheme, despite its one "drawback," is almost surprisingly independent of the space discretization except at the very early stage of the flow evolution. It has been observed, as usual, that finer grids produce better results.

All the above facts can be attributed to the reason that in the proposed approach the no-slip is satisfied explicitly to an accuracy of second order while solving both the vorticity transport and the Poisson equation. This has been made possible through the use of the new boundary condition (7) which can be said to be a numerical equivalent of Robbins condition. A new expression (10) has also been developed

TABLE III

Dependence on the Position of the Outer Boundary

r_∞	C_{dP}	C_{dF}	C_D
23.14	1.3018	0.8533	2.1552
43.37	1.2424	0.8215	2.0639
81.30	1.2191	0.8090	2.0281
152.40	1.2131	0.7958	2.0189

for evaluating the no-slip vorticity and used for solving (3). This expression is consistent with (7) and is second-order accurate.

Thus, the present work may be considered as a modest attempt to satisfy the no-slip in a better way within the framework of vorticity-stream function formulation using extremely simple finite-difference methods. Its success simply hints at the immense importance of the no-slip. It may not be wrong to conclude that algorithms which satisfy no-slip to a higher order of accuracy are likely to be more accurate and reliable.

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